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ON THE SUM OF ORDERS OF NON-CYCLIC AND NON-NORMAL SUBGROUPS IN A FINITE GROUP

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Abstract. Let G be a finite group and $\mathcal{C}(G)$ denote the set of all non-normal non-cyclic subgroups of G. In this paper, the function $\delta_c(G) = \frac{1}{|G|} \sum_{H \in \mathcal{C}(G)} |H|$ is introduced. In fact, we prove that, if $\delta_c(G) \leq \frac{10}{3}$, then either $G \cong A_5$, or G is solvable. We also find some examples of finite groups G with $\delta_c(G) \leq \frac{10}{3}$.

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1. Introduction

In this paper, all groups are assumed to be finite. Let $\mathcal{G}$ be the set of all groups of order n and $f : \mathcal{G} \longrightarrow \mathbb{R}$, where $\mathbb{R}$ is the real field. One may ask how the structure of G is influenced by some certain functions f. For example, T. De Medts and M. Tărnăuceanu [5] introduced the function

$$\sigma_1(G) = \frac{1}{|G|} \sum_{H \leq G} |H|.$$

Many results show that the arithmetical conditions of $\sigma_1(G)$ influence the solvability and supersolvability of G (see [8,10,13,14,15]). Similarly, W. Meng and J. Lu [11] only considered the sum of order of non-cyclic subgroups and introduced the function

$$\delta(G) = \frac{1}{|G|} \sum_{H \leq G} \{|H| \mid H \text{ is non-cyclic}\}.$$

They showed that if $\delta(G) < \frac{13}{3}$, then G is solvable, and if $\delta(G) < 1 + \frac{4}{|G|}$, then G is supersolvable. Furthermore, they gave a classification of finite groups with $\delta(G) \leq 2$.

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On the other hand, L. Cui et al. [4] considered the sum of order of non-normal subgroups. Consequently, they investigated the following function

$$\nu_0(G) = \frac{1}{|G|} \sum_{H \leq G, H \ntrianglelefteq G} |H|.$$

They proved that if $\nu_0(G) < \frac{29}{6}$, then G is solvable.

Inspired by above investigations, we consider the set of all non-cyclic and non-normal subgroups in a finite group. For conveniently, let $\mathcal{C}(G)$ denote the set of all non-cyclic and non-normal subgroups of G. The following function is defined.

$$\delta_c = \frac{1}{|G|} \sum_{H \in \mathcal{C}(G)} |H|.$$

It is easy to see that $\delta_c(G) = 0$ if and only if every non-cyclic subgroup of G is normal. Hence $\delta_c(G) = 0$ implies that G is a metahamiltonian group (i.e., every non-abelian subgroup of G is normal). The structure of metahamiltonian p-groups can be found in [1,3,6,7,9]. Thus, it seems to be interesting to study the properties of finite groups in terms of $\delta_c(G)$.

In this paper, we will prove the following result.

Theorem 1.1. *Let G be a group. If $\delta_c(G) \leq \frac{10}{3}$, then either $G \cong A_5$, or G is solvable.*

Lemma 2.6(2) shows that $\delta_c(A_5) = \frac{10}{3}$, therefore the bound in Theorem 1.1 is the best possible. Furthermore, we will find some finite groups G with $\delta_c(G) < \frac{10}{3}$ in Section 4. All unexplained notations and terminologies are standard and can be found in [12].

2. Preliminaries

In this section, we collect some results which will be used in the sequel.

Lemma 2.1. *Let G be a finite group and N be a normal subgroup of G. Then*

$$\delta_c(G/N) \leq \delta_c(G).$$

Proof. Let G be a finite group and N be a normal subgroup of G. We have

$$\begin{aligned} \delta_c(G/N) &= \frac{1}{|G/N|} \sum_{H/N \in \mathcal{C}(G/N)} |H/N| \\ &= \frac{1}{|G|} \sum_{H/N \in C(G/N)} |H| \\ &\leq \frac{1}{|G|} \sum_{H \in C(G)} |H| \\ &= \delta_c(G), \end{aligned}$$

as desired. □

Lemma 2.2. [10, Lemma 2.1] *Let G be a finite group and $[K]$ be the conjugacy class of a self-normalizing subgroup K of G. Then*

$$\textstyle\sum_{H\in[K]} |H| = |G|.$$

Lemma 2.3. [2, Theorem 2] *If a finite group G has at most 2 conjugacy classes of non-normal maximal subgroups, then G is solvable.*

Lemma 2.4. [2, Theorem 1] *Let G be a finite non-solvable group. Then G has three conjugacy classes of maximal subgroups if and only if either $G/\Phi(G) \cong PSL(2,7)$ or $PSL\,(2,2^p)$, where p is a prime.*

Lemma 2.5. [10, Lemma 2.4] *Let $p \geq 5$ be a prime, $G = PSL\,(2,2^p)$. Then*

$$\sum_{H\leq G,\ H\ non\text{-}cyclic} |H| \geq p|G|.$$

Lemma 2.6. *We have*

(1) $\delta_c(PSL(2,7)) > 5 > \frac{10}{3}$*;*
(2) $\delta_c(PSL(2,2^p)) > \frac{10}{3}$*, where p is a prime.*

Proof. (1) Let $G \cong PSL(2,7)$. Then G has exactly three classes of maximal subgroups, which are clearly neither cyclic nor normal. Furthermore, G has at least two conjugacy classes of non-cyclic second maximal subgroups which are isomorphic to S_3 and D_8, respectively. Obviously, S_3 and D_8 are self-normalizing second maximal subgroups of G. By Lemma 2.2, we have $\delta_c(G) > 5 > \frac{10}{3}$.

(2) Let $G \cong PSL\,(2,2^p)$, where p is a prime. If $p = 2$, then $G \cong A_5$. Now, noting that G has three conjugacy classes of maximal subgroups, says $[A_4]$, $[S_3]$ and $[D_{10}]$. Let $T \in Syl_2(G)$, then T is non-cyclic. So we have $\mathcal{C}(G) = \{[A_4],[S_3],[D_{10}],[T]\}$. It follows that $\delta_c(G) = \frac{1}{|G|}(3|G| + 5\times 4) = \frac{10}{3}$.

Suppose that $p \geq 3$. If $p \geq 5$, then $\delta_c(G) \geq p \geq 5 > \frac{10}{3}$ by Lemma 2.3. In the following, suppose that $p = 3$, then $G \cong PSL(2,8)$. It is well known that G has exact three conjugacy classes of maximal subgroups, i.e., $\left[M_1 \cong 2^3 : Z_7\right]$, $[M_2 \cong D_{18}]$ and $[M_3 \cong D_{14}]$. Furthermore, G possesses a conjugacy class of second maximal subgroups which is self-normalizing in G says $[S \cong D_6]$. Applying Lemma 2.2 again, we have $\delta_c(G) > \frac{1}{|G|}\left(\sum_{i=1}^3 \sum\limits_{H\in[M_i]} |H| + \sum\limits_{H\in[S]} |H|\right) = \frac{1}{|G|}(3|G| + |G|) = 4 > \frac{10}{3}$. □

3. The proof of Theorem 1.1

Proof. Suppose that G is a non-solvable finite group, which satisfies $\delta_c(G) \leq \frac{10}{3}$ and is not isomorphic to A_5, and suppose that G is of minimal order satisfying these conditions. Let N be a solvable normal subgroup of G. We have

$$\delta_c(G/N) \le \delta_c(G) \le \tfrac{10}{3}$$

by Lemma 2.1. If $N \neq 1$, then $|G/N| < |G|$ and hence G/N is solvable by the minimality of $|G|$. This implies that G is solvable, a contradiction. Therefore, $N = 1$. In particular, the Frattini subgroup $\Phi(G) = 1$.

First we show that G has exactly three conjugacy classes of non-normal maximal subgroups. Let $[M_1], [M_2], \cdots, [M_t]$ be the t conjugacy classes of non-normal maximal subgroups of G. Since G is non-solvable, it is well known that G has no abelian maximal subgroups. In particular, G has no cyclic maximal subgroups. Therefore, $\delta_c(G) \ge \frac{1}{|G|}\left(\sum\limits_{i=1}^{t} \sum\limits_{H\in[M_i]} |H|\right) = t$. By hypothesis, $\delta_c(G) \le \frac{10}{3}$ which leads to $t \le 3$. If $t \le 2$, then G is solvable by Lemma 2.3, a contradiction. Thus, $t = 3$, i.e., G has exactly three conjugacy classes of non-cyclic non-normal maximal subgroups.

Second, we show that G is not a simple group. Suppose that G is simple, then $G \cong PSL(2,7)$ or $PSL(2,2^p)$ by Lemma 2.4. Applying Lemma 2.6, we know that $\delta_c(G) \ge \frac{10}{3}$ if $p \ge 3$. This implies that $G \cong PSL(2,2^p) \cong A_5$. This is a contradiction again.

Hence G is a non-simple non-solvable group and there exists a non-trivial normal subgroup N of G. Consider the factor group G/N, then $1 < |G/N| < |G|$. Applying Lemma 2.1 again, we have $\delta_c(G/N) \le \delta_c(G) \le \frac{10}{3}$. By induction, G/N is solvable. Therefore, G has a normal maximal subgroup M and $|G/M|$ is a prime. Since G is non-solvable, also N is non-solvable. Let $S = \bigcap\{N \mid N \trianglelefteq G \text{ and } G/N \text{ is solvable}\}$ be the solvable residual of G. Then S is non-solvable and it is the minimal normal subgroup of G with G/S solvable. Let S' be the derived subgroup of S, then $S = S'$ (Otherwise, if $S' < S$, then G/S' would be solvable, a contradiction).

In the following, we claim that $N_G(L)$ is a self-normalizing maximal subgroup of G for every maximal subgroup L of S. It is easily seen that $S = S'$ implies that L is non-normal in S. Thus, if $g \notin N_G(L)$ for some $g \in N_G(N_G(L))$, then $L^g \neq L$. This obliges to $L \trianglelefteq \langle L, L^g\rangle = S$ which is a contradiction. So $g \in N_G(L)$. Moreover, applying Lemma 2.2, we have $\sum\limits_{H\in[N_G(L)]} |H| = |G|$. Hence if $[N_G(L)] \neq [M_i]$ for $i = 1,2,3$, then $\delta_c(G) \ge 4$. This is a contradiction. So $N_G(L)$ is a maximal subgroup of G.

Now, we shall show that S has exactly three conjugacy classes of maximal subgroups. Suppose that S has at least four conjugacy classes of maximal subgroups, say $[L_1], [L_2], [L_3]$ and $[L_4]$. If $N_G(L_i)$ is not conjugate to $N_G(L_j)$ for any $i \neq j$, then there exist four conjugacy classes of self-normalizing maximal subgroups

$$N_G(L_1), N_G(L_2), N_G(L_3) \text{ and } N_G(L_4)$$

of G which contradict to $t = 3$. Thus, at least two of $N_G(L_1), N_G(L_2), N_G(L_3)$ and $N_G(L_4)$, say $N_G(L_1)$ and $N_G(L_2)$ are conjugate in G. So there exists some $g \in G$ such that $N_G(L_1)^g = N_G(L_2)$. If $L_1^g \neq L_2$, then L_2 is normal in $\langle L_2, L_1^g \rangle = S$, a contradiction. So we have $L_2 = L_1^g$. Observe that $S \not\leq N_G(L_1)$, we get that $G = N_G(L_1) S$ and $g = ns$, with $n \in N_G(L_1)$ and $s \in S$. This implies that $L_2 = L_1^g = L_1^{ns} = L_1^s$, i.e., L_1 and L_2 are conjugate in S, a contradiction. So S has at most three conjugacy classes of maximal subgroups.

Observe that S is non-solvable, we know that $S/\Phi(S) \cong PSL(2,7)$ or $PSL(2, 2^p)$ by Lemma 2.6. Since $\Phi(S) \leq \Phi(G) = 1$, we have $S \cong PSL(2,7)$ or $PSL(2, 2^p)$. Therefore, $C_G(S) \cap S = Z(S) = 1$. This implies that $C_G(S) \cong SC_G(S)/S \leq G/S$ is solvable. As G has no non-trivial solvable normal subgroups, we get that $C_G(S) = 1$. So we have $G \cong G/C_G(S) \leq \mathrm{Aut}(S)$.

By above arguments, we know that S contains exactly three conjugacy classes of self-normalizing non-cyclic maximal subgroups, say $[L_1], [L_2]$ and $[L_3]$, and these subgroups are non-normal in S. Applying Lemma 2.2 again, we have $\sum_{H \in [L_i]} |H| = |S|$ for any $i \in \{1, 2, 3\}$. We get that

$$\delta_c(G) \geq \frac{1}{|G|} \left(\sum_{i=1}^{3} \sum_{H \in [M_i]} |H| + \sum_{i=1}^{3} \sum_{H \in [L_i]} |H| \right) = \frac{1}{|G|}(3|G| + 3|S|) = 3 + \frac{3|S|}{|G|}.$$

If $S \cong PSL(2,7)$, then $|\mathrm{Aut}(S)| = 2|PSL(2,7)|$ (see [12, 8.8 in chapter 6]) which implies that $|G| = 2|S|$. Thus $\delta_c(G) \geq \frac{1}{|G|}(3|G| + 3|S|) = \frac{9}{2} > \frac{10}{3}$. This is a contradiction.

Suppose that $S \cong PSL(2, 2^p)$, then $|Aut(S)| = p|S|$ (see [12, 8.8 in chapter 6]) and hence $|G| = p|S|$. If $p = 2$, or 3, then we have

$$\delta_c(G) \geq \tfrac{1}{|G|}(3|G| + 3|S|) = 3 + \tfrac{3}{p} \geq 4 > \tfrac{10}{3},$$

which is another contradiction. In the following, we suppose that $p \geq 5$. Observe that every proper subgroup of S is solvable. We know that every non-trivial subgroup of S is non-normal in G. So we can consider all non-cyclic proper subgroups of S. Applying Lemma 2.5, we have $\sum\limits_{H < S, H \text{ non-cyclic}} |H| \geq (p-1)|S|$. It follows that

$$\delta_c(G) \geq \frac{1}{|G|} \left(\sum_{i=1}^{3} \sum_{H \in [M_i]} |H| + \sum_{H < S, H \text{ non-cyclic}} |H| \right) \geq \frac{1}{|G|}(3|G| + (p-1)|S|)$$
$$= 3 + \frac{(p-1)|S|}{|G|} = 4 - \frac{1}{p} \geq 4 - \frac{1}{5} > \frac{10}{3}.$$

This is the final contradiction. The proof of theorem is complete. □

4. Several families of finite groups with $\delta_c < \frac{10}{3}$

In this section, we first look for the δ_c of some important classes of groups, eventually focusing on some groups which have small δ_c.

Proposition 4.1. *Let $G \cong D_{2^n}$ be the dihedral group of order 2^n, where $n \geq 3$. Then $\delta_c(G) = n - 3$.*

Proof. Let $G = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, a^b = a^{-1} \rangle$. If $n = 3$, then $G \cong D_8$. It is easy to see that $\delta_c(G) = 0$. Thus, the conclusion holds. Suppose that $n \geq 4$. By the defining relations of G, we can find that all non-cyclic non-normal subgroups of G are as follows

$$\langle a^{2^k}, a^l b \rangle, \text{ where } 2 \leq k \leq n-2 \text{ and } 0 \leq l \leq 2^k - 1.$$

Observe that $|\langle a^{2^k}, a^l b \rangle| = 2^{n-k}$. It follows that

$$\delta_c(G) = \frac{1}{2^n} \sum_{k=2}^{n-2} \sum_{l=0}^{2^k-1} |\langle a^{2^k}, a^l b \rangle| = \frac{1}{2^n} \sum_{k=2}^{n-2} 2^k \cdot 2^{n-k} = n - 3.$$

So the conclusion holds. □

Proposition 4.2. *Let $G \cong Q_{2^n}$ be the generalized quaternion group of order 2^n, where $n \geq 4$. Then $\delta_c(G) = n - 4$.*

Proof. Let $G = \langle a, b \mid a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, a^b = a^{-1} \rangle$. Then G contains a unique involution $t = a^{2^{n-2}}$ and $G/\langle t \rangle \cong D_{2^{n-1}}$. So we get $\delta_c(G) = \delta_c(G/\langle t \rangle) = \delta_c(D_{2^{n-1}}) = n - 4$. □

Proposition 4.3. *Let $G \cong D_{2p^m}$ be the dihedral group of order $2p^m$, where p is an odd prime and $m \geq 1$. Then $\delta_c(G) = m - 1$.*

Proof. Let $G = \langle a, b \mid a^{p^m} = b^2 = 1, a^b = a^{-1} \rangle$. If $m = 1$, then $G \cong D_{2p}$. It is easily seen that $\delta_c(G) = 0$. Thus, the conclusion holds. Suppose that $m \geq 2$. By the defining relations of G, we can find that all non-cyclic non-normal subgroups of G are as follows $\langle a^{p^k}, a^l b \rangle$, where $1 \leq k \leq m-1$, $0 \leq l \leq p^k - 1$.
Observe that $|\langle a^{p^k}, a^l b \rangle| = 2p^{m-k}$. So we have

$$\delta_c(G) = \frac{1}{2 \cdot p^m} \sum_{k=1}^{m-1} \sum_{l=0}^{p^k-1} |\langle a^{p^k}, a^l b \rangle| = \frac{1}{2 \cdot p^m} \sum_{k=1}^{m-1} p^k \cdot (2 \cdot p^{m-k}) = m - 1.$$

So the conclusion holds. □

Proposition 4.4. *Let $m = p_1 p_2 \cdots p_s$ and $G \cong D_{2m}$ the dihedral group of order $2m$, where $p_1, \ldots, p_s$ are distinct odd primes. Then $\delta_c(G) = 2^s - 2$.*

Proof. Let $G = \langle a, b \mid a^m = b^2 = 1, a^b = a^{-1} \rangle$. If $s = 1$, then $G \cong D_{2p_1}$. It is easily seen that $\delta_c(G) = 0$. Thus, the conclusion holds. Suppose that $s \geq 2$. For any subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, s\}$, where $1 \leq k \leq s-1$, set

$$H_{i_1 i_2 \ldots i_k} = \langle a^{p_{i_1} \cdots p_{i_k}}, b \rangle.$$

Then each $H_{i_1 i_2 \ldots i_k} \cong D_{2m/p_{i_1} \cdots p_{i_k}}$ is a self-normalizing subgroup of G by the defining relations of G. By Lemma 2.2, we get $\sum\limits_{H \in [H_{i_1 i_2 \ldots i_k}]} |H| = |G|$. It follows that

$$\delta_c(G) = \frac{1}{|G|} \sum_{\{i_1, \ldots, i_k\} \subset \{1,2,\ldots,s\}} \sum_{H \in [H_{i_1 i_2 \ldots i_k}]} |H| =$$

$$= \frac{1}{|G|} \sum_{\{i_1, \ldots, i_k\} \subset \{1,2,\ldots,s\}} |G| = \binom{s}{1} + \binom{s}{2} + \cdots + \binom{s}{s-1} = 2^s - 2,$$

as desired. □

Proposition 4.5. *Let $G \cong M_{p^n} = \langle a, b \mid a^{p^{n-1}} = b^p = 1, a^b = a^{-1+p^{n-2}} \rangle$, where p is an odd prime and $n \geq 4$. Then $\delta_c(G) = \frac{p^{n-3}-1}{p^{n-2}(p-1)} < 1$.*

Proof. Let $G = M_{p^n}$. Then G possesses a unique non-cyclic subgroup $\langle a^{p^{n-\lambda}}, b \rangle$ of order p^λ for any $2 \leq \lambda \leq n$. Observe that $\langle a, b \rangle$ and $\langle a^p, b \rangle$ are normal in G, so we get $\delta_c(G) = \frac{p^2 + \cdots + p^{n-2}}{p^n} = \frac{p^{n-3}-1}{p^{n-2}(p-1)} < 1$. So the proof is completed. □

Proposition 4.6. *Let $G \cong S_4$. Then $\delta_c(G) = \frac{5}{2}$.*

Proof. Suppose $G \cong S_4$, then G contains two conjugacy classes of non-normal maximal subgroups, that is, $[D_8]$ and $[S_3]$. Furthermore, G has a conjugacy class of non-normal subgroups $[V_4]$ of order 4, where $V_4 \cong Z_2 \times Z_2$ is non-cyclic and $N_G(V_4) \cong D_8$. So $\delta_c(G) = \frac{24+24+4 \cdot 3}{24} = \frac{5}{2}$. So the conclusion holds. □

By Propositions 4.1-4.6, we can find some finite groups with $\delta_c(G) < \frac{10}{3}$. Hence, the following result is immediate.

Theorem 4.7. *Suppose that G is one of the groups $D_{2^n} (n \leq 6)$, $Q_{2^n} (n \leq 7)$, $D_{2p^n} (n \leq 4)$, D_{2pq}, M_{p^n} or S_4. Then $\delta_c(G) < \frac{10}{3}$.*

It seems meaningful to determine the structure of finite groups G with $\delta_c(G) \leq \frac{10}{3}$, so we have the following problem.

Problem 4.8. *Find all finite groups G with $\delta_c(G) \leq \frac{10}{3}$.*

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References

[1] L. An and Q. Zhang, *Finite metahamiltonian p-groups*, J. Algebra, 442 (2015), 23-35.

[2] V. A. Belonogov, *Finite groups with three classes of maximal subgroups*, Math. USSR-Sb., 59(1) (1988), 223-236.

[3] M. Brescia, M. Ferrara and M. Trombetti, *The structure of metahamiltonian groups*, Jpn. J. Math., 18(1) (2023), 1-65.

[4] L. Cui, W. Meng, J. Lu and W. Zheng, *A new criterion for solvability of a finite group by the sum of orders of non-normal subgroups*, Colloq. Math., 174(2) (2023), 169-176.

[5] T. De Medts and M. Tărnăuceanu, *Finite groups determined by an inequality of the orders of their subgroups*, Bull. Belg. Math. Soc. Simon Stevin, 15(4) (2008), 699-704.

[6] R. Dedekind, *Ueber Gruppen, deren sämmtliche Theiler Normaltheiler sind*, Math. Ann., 48 (1897), 548-561.

[7] X. Fang and L. An, *A classification of finite metahamiltonian p-groups*, Commun. Math. Stat., 9(2) (2021), 239-260.

[8] M. Garonzi and M. Patassini, *Inequalities detecting structural properties of a finite group*, Comm. Algebra, 45(2) (2017), 677-687.

[9] I. N. Herstein, *A remark on finite groups*, Proc. Amer. Math. Soc, 9(2) (1958), 255-257.

[10] M. Herzog, P. Longobardi and M. Maj, *On a criterion for solvability of a finite group*, Comm. Algebra, 49(5) (2021), 2234-2240.

[11] W. Meng and J. Lu, *On the sum of non-cyclic subgroups order in a finite group,* Comm. Algebra, 52(3) (2024), 1084-1096.

[12] M. Suzuki, Group Theory II, Fundamental Principles of Mathematical Sciences, 248, Springer-Verlag, New York, 1986.

[13] M. Tărnăuceanu, *Finite groups determined by an inequality of the orders of their subgroups II*, Comm. Algebra, 45(11) (2017), 4865-4868.

[14] M. Tărnăuceanu, *On the solvability of a finite group by the sum of subgroup orders*, Bull. Korean Math. Soc., 57 (2020), 1475-1479.

[15] M. Tărnăuceanu, *On the supersolvability of a finite group by the sum of subgroup orders*, J. Algebra Appl., 21 (2022), 2250232 (7 pp).

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